Harmonic Oscillation Lab

Simple Harmonic Oscillation (SHO) stands high among the most important problems in all of Physics. It is a "Universal Problem" so to speak. What's particularly impressive is that the profusion of settings where we find it often ***look*** so very different from each other … and yet they ***behave*** the same. That's the idea behind the phrase "*Universal Form*". Buried inside each of these systems is a pattern … a universal pattern. Once we recognize the pattern we can immediately "read off" the "universal consequences" that follow from that pattern. We don't have to solve the problem repeatedly. This is really just an amplification of what we did before. If before we realized that solving a problem symbolically allowed us to solve "an infinity" of problems in one fell swoop, well … now we're just amplifying that by solving an *infinity of infinities* all at once. We're studying mechanics in this course – but the "pattern" will appear in electrodynamics, thermodynamics, quantum mechanics, acoustics and fluid dynamics, cosmology … even the stock market! Everywhere!

There are many ways to recognize the pattern but since this is mechanics we will start from Newton's Second Law.

$$m(d^2\vec{r}/\mathrm{d}t^2) = \vec{F}$$

Many a young student is not really aware that in writing this down Newton was also inventing the world's first "Differential Equation". The whole idea is to start here and find out what the variable called $\vec{r}$ is doing as time progresses. How does it behave ? What's its dependence on time ? This depends crucially (though not exclusively) on the force and all its details.
As we will discover, if the force is a "linear restoring force" … then we will see SHO as the solution. The bare mathematical equation we are looking for has the following "form" or "shape" :

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

In this equation the quantity "*x*" doesn't have to be a length. In fact it can be any kind of variable at all. It can be an angle, an electric field strength, a density, a temperature … *any* variable at all. It is clearly an "acceleration" of something and, of course, in mechanics comes from the left hand side of Newton's 2nd Law. The right hand side comes from the right hand side of Newton's 2nd Law and has several characteristic features. First, the quantity ω^2 is just a constant number and will be built out of the constants of the problem. We have expressed it as a square to emphasize that it must be positive and then the minus sign in front makes for a definite negative overall sign. Finally, the variable *x* appears once more … linearly. This is a ***crucial*** observation. So these are the three things we look for :

1) a minus sign,
2) a positive constant,
3) the coordinate again to the first power.

If that is what we have to start with, then the solution follows by an inescapable chain of reasoning as:

$$x(t) = A \cdot \sin(\omega t + \varphi_o)$$

The constant numbers A and φ_o are called the amplitude and the phase shift and are chosen to fit the specific conditions of the problem. Any values whatsoever of A and φ_o still leave us with a solution which obeys the defining equation. For those of you with a little more experience, they are constants of integration and come from "integrating the defining equation twice".

It's important to understand that an equation and its solution are exactly the "same information" rearranged differently. That's the miracle of mathematics. A few things are important to notice:

1) We could have used cosine instead of sine since these differ only by a "phase shift" … and we can make the constant φ_o anything we want and still satisfy the equation of motion.

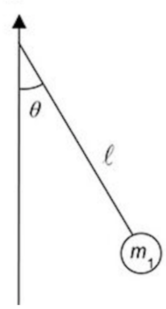
2) The constant ω appeared in the defining equation and by checking dimensions clearly has to be an "inverse time". Since the dominant feature of "oscillation" is to repeat at regular time intervals called the period "*T*" … and since the trig functions repeat whenever the argument increases by 2π , we must have $\omega T = 2\pi$ or equivalently $\omega = 2\pi/T$. Clearly, ω has the meaning of 2π radians of angle every *T* worth of time. Therefore ω is indeed the angular velocity.

3) Of particular importance is the observation that the period *T* (or equivalently ω) has nothing to do with *A*, the amplitude. No matter how large the swing is, the amount of time it takes to do it doesn't change. This is counter-intuitive and very surprising to most people.

The Lab

This lab will examine two systems: 1) a simple pendulum, and 2) a mass hanging from a simple spring. In each case you will take measurements successively by varying some feature of the system. In the first case you will check that our predictions are upheld. In the second case you will simply assume that the system truly is a SHO and ***use*** the properties that go with that to discover an amazing property of real springs. You will discover that real springs have a "magic fraction" associated with themselves. This magic fraction is such that, if you take the total mass of the real spring and multiply it times the "magic fraction" and then "give that mass" to the mass hanging on the spring (and that it is accelerating), then the actual spring may be treated as "ideal and massless" from that point on. This is a great way to make our real springs into *ideal springs*.

The Simple Pendulum



A mass m is swinging on the end of a string of length ℓ . The mass is actually executing motion along a circular arc. The proper descriptive equation is thus the equation of circular motion: $I\alpha = \tau$. In this case, the moment of inertia $I = ml^2$ and the torque $\tau = -mgl \cdot \sin(\theta)$. If we measure our angles in radians and keep the angle "smallish" , i.e. $\theta \ll 1$ (remember, 1 radian is nearly 60 degrees so this is scarcely much of a restriction), then we may use the "small angle approximation", $\sin(\theta) \approx \theta$. Our equation of motion is well approximated as :

$$ml^2 \frac{d^2\theta}{dt^2} = -mgl\theta$$

Canceling common factors yields:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

If we compare this with our SHO prototype equation, we see that the "constant cluster" $\frac{g}{l}$ stands in the place where we put the symbol ω^2 and, indeed, has the right dimensionality. We conclude that it must follow:

$$\frac{2\pi}{T} = \sqrt{\frac{g}{l}}$$

Equivalently we may write:

$$T^2 = \frac{(2\pi)^2}{g} \cdot l$$

This is the main result for our experiment. It shows the dependence of the period on the string length, the mass (**no** dependence !), and of course no dependence on the starting angle. You will verify these properties next.

Experiment 1.

Vary the mass and *graph the outcome of T vs. m* . The pendulum has l = .97 meter.

Material	**Mass kg**	**T seconds**
Aluminum	0.239	1.947
Steel	0.667	2.0
Bronze	0.707	1.987
Copper	0.750	1.985
Lead	0.908	1.974

Experiment 2.

Vary the string length and *graph the outcome of T^2 vs. l* . We then expect a straight line and the slope to be $\frac{(2\pi)^2}{g}$. Once you have your graph, verify that it is linear, and then find the best straight-line-fit and compare your slope to the prediction.

Trial	l **meters**	**T seconds**
1	0.2	0.840
2	0.4	1.276
3	0.6	1.583
4	0.8	1.818
5	1.0	2.051

Experiment 3.

Vary the starting angle θ_o and *graph the outcome of T vs. θ_o* . What do you expect? What do you see ? At larger angles has the period altered in any noticeable way? If the period really *were* constant, then $T_k/T_1 = 1$ would be true for each trial T_k . Attempt to see the discrepancy at larger angles by *graphing* ($T_k/T_1 - 1$) versus the ***angle in radians***. Attempt to *curve fit your graph to a power law*: i.e. use the form $C\,\theta^\gamma$ where the constants C and γ are to be found by curve fitting this function so that it sits on top of your numbers ($T_k/T_1 - 1$). This is a classical problem and the constants are predicted in our next level mechanics class. You are determining them experimentally. The idea here is that we made a restriction on the validity of our model: "we were to keep the angles small". If we violate that restriction, then we must expect that deviations from our "ideal" predictions may start to appear. A more sophisticated model will include these deviations too.

Trial	θ_o	**T seconds**
1	15°	1.353
2	30°	1.351
3	45°	1.382
4	60°	1.404
5	75°	1.454

Mass on a Spring

A mass m is hanging at rest from a real spring . The rest-stretch of the spring cancels gravity which then plays no further role. The resting spot now plays the role of our origin of coordinates. If we now pull the mass down yet further and release, the equation of motion is:

$$m\frac{d^2x}{dt^2} = -kx$$

Consolidating factors yields:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If we compare this with our SHO prototype equation, we see that the "constant cluster" $\frac{k}{m}$ stands in the place where we put the symbol ω^2 and, indeed, has the right dimensionality. We conclude that it must logically follow:

$$\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

Equivalently we may write:

$$T^2 = \frac{(2\pi)^2}{k} \cdot m$$

This is the main result for our experiment. It shows the dependence of the period on the moving mass and the spring constant. We observe, of course, no dependence on the initial displacement.

Experimental Data.

Vary the added mass and *graph the outcome of T^2 vs. m_{added}* . You expect a straight line! Does that happen?

Trial	Added Mass kg	T seconds
1	0.05	0.704
2	0.1	0.805
3	0.15	0.962
4	0.2	1.066
5	0.25	1.189

We now reason as follows. The spring itself has a total mass of 0.1665 kg … and this is substantial! That mass is moving when the system oscillates …but not all of itself is moving the same way! The top of the spring isn't moving at all. The bottom of the spring is moving right along with the added mass. We expect the spring to contribute "some part of itself" to the moving mass … but just how much? This will be the "magic fraction" which is true for all springs. If we express the mass in our fundamental relation as the sum of two parts, namely : $m = m_{\mathbf{added}} + m_{\mathbf{(spring\ fraction)}}$, then we expect:

$$T^2 = \frac{(2\pi)^2}{k} \cdot (m_{\mathbf{added}} + m_{\mathbf{(spring\ fraction)}})$$

If we place T^2 on the vertical axis and $m_{\mathbf{added}}$ on the horizontal axis, then this relation predicts that we should see the "form" y = (slope) x + (intercept) , where the slope is given by

$$slope = \frac{(2\pi)^2}{k}$$

And the intercept is given by:

$$intercept = \frac{(2\pi)^2}{k} \cdot m_{\mathbf{(spring\ fraction)}}$$

So perform ***a straight line curve fit to your*** T^2 *vs.* m_{added} graph. Next take the ratio of (intercept)/(slope) and that should give you $m_{\mathbf{(spring\ fraction)}}$! Finally, take the ratio

$$m_{\mathbf{(spring\ fraction)}}/(total\ spring\ mass)$$

And this yields the "magic fraction". It should be a simple fraction. You won't hit it *exactly* … but you'll be close. ***What do you find*** ?